On the gravitational field induced by static electromagnetic sources

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It is argued that static electric or magnetic sources induce Weyl-Majumdar-Papapetrou solutions for the metric of spacetime. Their gravitational acceleration includes a term many orders of magnitude stronger than usual perturbative terms. It gives rise to a number of effects, which can be detected experimentally. Two electrostatic and two magnetostatic examples of physical set-ups with simple symmetries are proposed. The different ways in which mass sources enter and complicate the pure electromagnetic picture are described.

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I. INTRODUCTION

In classical Newton-Maxwell physics the electromagnetic (EM) fields have no influence upon gravity, which is generated by sources of mass. In general relativity EM fields alter the metric of spacetime and induce a gravitational force through their energy-momentum tensor

$$T^{\mu}_{\nu} = \frac{1}{4\pi}\left(F^{\mu\alpha}F_{\nu\alpha} - \frac{1}{4}\delta^{\mu}_{\nu}F^{\alpha\beta}F_{\alpha\beta}\right), \tag{1}$$

where

$$F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu} \tag{2}$$

is the electromagnetic tensor and A_{μ} is the four-potential. T^{μ}_{ν} enters the r.h.s. of the Einstein equations

$$R^{\mu}_{\nu} = \kappa T^{\mu}_{\nu}. \tag{3}$$

We have taken into account that $T^{\mu}_{\mu} = 0$. The Einstein constant is

$$\kappa = \frac{8\pi G}{c^4} = 2.07\times 10^{-48} s^2/cm.g, \tag{4}$$

where $G = 6.674\times 10^{-8} cm^3/g.s^2$ is the Newton constant and $c = 2.998\times 10^{10} cm/s$ is the speed of light. We shall use the Gauss system (CGS) of nonrelativistic units and occasionally the international system of practical units (SI).

In addition, the Maxwell equations are coupled to gravity through the covariant derivatives of $F_{\mu\nu}$

$$F^{\mu\nu}{}_{;\nu} = \frac{1}{\sqrt{-g}}\left(\sqrt{-g}F^{\mu\nu}\right)_{\nu} = -\frac{4\pi}{c}J_{\mu}, \qquad J^{\mu} = \sigma c u^{\mu}. \tag{5}$$

Here g is the metric's determinant, usual derivatives are denoted by subscripts, J^{μ} is the four-current, $u^{\mu} = dx^{\mu}/ds$ is the four-velocity of the charged particles with charge density σ. We shall study mainly electrovacuum solutions with $\sigma \neq 0$ only on some surface, specifying the boundary conditions. The Einstein-Maxwell equations (3,5) show how the EM-field leaves its imprint on the metric, which has to satisfy the Rainich conditions [1, 2, 3].

The gravitational force acting on a test particle is represented by the four-acceleration

$$g_{\mu} = c^2\frac{du_{\mu}}{ds} = c^2\Gamma_{\alpha,\mu\beta}u^{\alpha}u^{\beta} = \frac{c^2}{2}g_{\alpha\beta,\mu}u^{\alpha}u^{\beta}, \tag{6}$$

where $\Gamma^{\alpha}_{\mu\beta}$ are the Christoffel symbols. When the particle is at rest, $u^0 = (g_{00})^{-1/2}$ and

$$g_{\mu} = \frac{c^2}{2}\left(\ln g_{00}\right)_{\mu} \tag{7}$$

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for an arbitrary metric $g_{\alpha\beta}$.

In this paper we investigate the problem whether EM-fields can induce strong enough acceleration, rising above the gravimeter's threshold of $10^{-6}cm/s^2$ or even comparable to the mean Earth acceleration $g_e = 980.665cm/s^2$. Eqs. (3,4) show that the metric will be very near to the flat one without any singularities and faraway from the metric of a black hole. The question is whether the 20 orders of magnitude supplied by $c^2/2$ in Eq. (7) are enough to lift the EM-gravitational force to that of the Newtonian gravity of very massive bodies. In fact, we should consider the contravariant physical (tetrad) four-vector $g^{(\mu)} = \eta^{\mu\nu} g_{(\nu)}$, where $\eta_{\mu\nu} = diag\,(1, -1, -1, -1)$, but in cartesian coordinates and for such an almost flat metric it is indistinguishable from g_μ or $g_{(\mu)}$ except for a sign change.

It seems natural to use perturbation theory in the harmonic gauge where

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \qquad \Delta h_{\mu\nu} = -2\kappa T_{\mu\nu} \tag{8}$$

and the Poisson equation shows that $h_{\mu\nu}$ is extremely small, while $g_\mu \sim c^2\kappa = 1.85 \times 10^{-25} CGS$. It appears that the problem has been solved and the effect is negligible. However, there is a loophole, which is discussed in the sequel.

In Sec. II it is shown that the gravitational acceleration in Weyl-Majumdar-Papapetrou (WMP) fields [4, 5, 6, 7, 8] includes a term proportional to $\sqrt{\kappa}$ and linear in the fields. In Sec. III axially-symmetric static Einstein-Maxwell fields are revisited. The three classes of Weyl solutions are described, as well as their relation to the general solution and situations where they become the most general solution. It is argued that charge and current distributions determine their harmonic master-potential and induce them as pure electromagnetic effect upon the spacetime metric. A general formula is given for the acceleration and the question of hidden mass sources is investigated. In Sec. IV a short review is given of WMP fields in the general static case and in the presence of charged dust or perfect fluid.

In Sec. V we give two electrostatic examples of charged surfaces, which induce Weyl solutions. They involve plane, spherical and spheroidal symmetry. The charged plane and the Reissner-Nordström (RN) solution are discussed from a Weyl point of view. The peculiarities in the junction conditions for Weyl fields are pointed out. Different physical issues, such as the force arising inside a charged capacitor and repulsive gravity around a charged sphere are explored. In Sec. VI we give two magnetostatic examples, a current loop and an open solenoid. In Sec. VII the main results about WMP fields are summarised. The last section contains a short discussion.

II. ROOT GRAVITY

Let us assume that the metric and the EM-fields do not depend on time. In this stationary case let us further simplify the problem by setting $A_\mu = \left(\bar{\phi}, 0, 0, 0\right)$. There is just an electric field

$$E_\mu = F_{0\mu} = -\bar{\phi}_\mu. \tag{9}$$

Obviously, T^μ_ν from Eq. (1) contains only quadratic terms in $\bar{\phi}_\mu$. This allows to hide κ from Eq. (3) by normalizing the electric potential to a dimensionless quantity

$$\phi = \sqrt{\frac{\kappa}{8\pi}}\bar{\phi}. \tag{10}$$

The factor 8π is chosen for future convenience. We shall see that this is a much more elegant way to get rid of the constants in the Einstein-Maxwell equations than the choice of relativistic units $c = 1$ and $G = 1$ ($8\pi G = 1$ sometimes).

Imagine now that in some exact solution g_μ is proportional to the electric field, contrary to the quadratic dependence in Eq. (8)

$$g_\mu = ac^2\phi_\mu = ac^2\sqrt{\frac{\kappa}{8\pi}}\bar{\phi}_\mu, \tag{11}$$

where a is some slowly changing function of order $O\left(1\right)$. For example, when $a = a\left(g_{00}\right)$ then Eqs. (7,11) lead to the functional dependence $f \equiv g_{00} = F\left(\phi\right)$. Let us further assume that the spacetime is static. Then the above functional dependence has the unique form [5, 6, 7]

$$f = 1 + B\phi + \phi^2. \tag{12}$$

In the axially-symmetric case this equation was found by Weyl already in 1917 [8] and such solutions are known as Weyl fields. The potential in Eq. (10) is very small everywhere and naturally goes to zero at infinity. Then asymptotic flatness fixes the first term which is otherwise an arbitrary constant. It is also fixed by the requirement to go back to

Minkowski spacetime when $\phi = 0$ since we are studying the EM-effect on gravity with no masses present. Later we will give arguments that the typical value of B is 2, so that the linear term in Eq. (12) is really present.

Thus in Weyl-Majumdar-Papapetrou fields we have

$$g_\mu = c^2 f^{-1} \left(\frac{B}{2} \sqrt{\frac{\kappa}{8\pi}} \bar{\phi}_\mu + \frac{\kappa}{8\pi} \bar{\phi} \bar{\phi}_\mu \right). \tag{13}$$

The first term is of the type given in Eq. (11), the second resembles the expression in Eq. (8). Let us note that

$$c^2 \sqrt{\frac{\kappa}{8\pi}} = \sqrt{G} = 2.58 \times 10^{-4}, \qquad c^2 \frac{\kappa}{8\pi} = \frac{G}{c^2} = 7.37 \times 10^{-27}. \tag{14}$$

Due to the square root, the first coefficient is 10^{23} times bigger than the second. We shall call gravitational fields, which have acceleration terms $\sim \sqrt{\kappa}$, root gravity. The WMP fields are an example, but there are others too. Thus general relativity has a Newtonian limit in the case of mass sources, where $g_\mu \sim G$ and a Maxwellian limit in the case of EM-sources, where $g_\mu \sim \sqrt{G}$. In relativistic units this effect is not seen. When $G = c = 1$ then $\sqrt{\kappa/8\pi} = \kappa/8\pi = 1$ and there is no difference between root and usual gravity. When $8\pi G = c = 1$ then $\kappa = 1$ and $\sqrt{\kappa/8\pi} = 0.2$, while $\kappa/8\pi = 0.04$. The difference is just an order of magnitude.

Provided that $B = 2$ our search for a strong gravitational acceleration induced by EM-fields doesn't seem so doomed as in the introduction. Electrostatic generators can create potential differences of six million volts or

$$\bar{\phi}_{\max} = 2 \times 10^4 CGS. \tag{15}$$

If it were applied to a capacitor with distance of $1cm$ between the plates, the electric field will be of the same order. It compensates the root coefficient in Eq. (15) and we get acceleration of about $1cm/s^2$, which is perfectly measurable. Static magnetic fields create the same gravitational effects as static electric fields [9, 10, 11], so a field of $33.2T$ may induce in principle acceleration of about $10cm/s^2$. One needs just two orders more to counter Earth's gravity. These effects are much stronger than any other known general relativistic effects, including gravitational waves and gravitomagnetism, which are currently under intensive study. They may be produced in a finite region in a laboratory if we learn how to create WMP fields. Therefore, in the next sections we revisit these fields, putting the emphasis on their physical applicability.

III. WEYL FIELDS REVISITED

Let us start with the axially-symmetric static metric

$$ds^2 = f \left(dx^0\right)^2 - f^{-1} \left[e^{2k} \left(dr^2 + dz^2\right) + r^2 d\varphi^2\right], \tag{16}$$

where $x^0 = ct$, $x^1 = \varphi$, $x^2 = r$, $x^3 = z$ are cylindrical coordinates, $f = e^{2u}$ and u is the first, while k is the second gravitational potential. Both of them depend only on r and z. Let $A_\mu = \left(\bar{\phi}, \bar{\chi}, 0, 0\right)$ where $\bar{\chi}$ is the true magnetic potential. Following Tauber [12] we introduce the auxiliary potential $\bar{\lambda}$

$$\lambda_r = \frac{f}{r} \chi_z, \qquad \lambda_z = -\frac{f}{r} \chi_r, \tag{17}$$

so that

$$F^{\varphi r} = \frac{f e^{-2k}}{r} \bar{\lambda}_z, \qquad F^{z\varphi} = \frac{f e^{-2k}}{r} \bar{\lambda}_r \tag{18}$$

describe the axial and the radial components of the magnetic field. For the electric field one has

$$E_r = F_{0r} = -\bar{\phi}_r, \qquad E_z = F_{0z} = -\bar{\phi}_z. \tag{19}$$

The field equations read

$$\Delta u = e^{-2u} \left(\phi_r^2 + \phi_z^2 + \lambda_r^2 + \lambda_z^2\right), \tag{20}$$

$$\Delta \phi = 2 \left(u_r \phi_r + u_z \phi_z\right), \tag{21}$$

$$\Delta\lambda = 2\left(u_r\lambda_r + u_z\lambda_z\right), \tag{22}$$

$$\phi_r\lambda_z = \phi_z\lambda_r, \tag{23}$$

$$\frac{k_r}{r} = u_r^2 - u_z^2 - e^{-2u}\left(\phi_r^2 - \phi_z^2 + \lambda_r^2 - \lambda_z^2\right), \tag{24}$$

$$\frac{k_z}{r} = 2u_r u_z - 2e^{-2u}\left(\phi_r\phi_z + \lambda_r\lambda_z\right), \tag{25}$$

$$k_{rr} + k_{zz} = \Delta u - \left(u_r^2 + u_z^2\right), \tag{26}$$

where $\Delta = \partial_{rr} + \partial_{zz} + \partial_r/r$. We have used the definition given in Eq. (10) and a similar one for λ. Using Eq. (23) one can prove that $\lambda = \lambda(\phi)$ and the dependence is linear [9, 10]. This result holds also for a general static metric [11]. It is enough to engage just an electric field, there being a trivial magnetovac analogue to every electrovac solution. Eqs. (20-23) reduce to [13]

$$\Delta u = e^{-2u}\left(\phi_r^2 + \phi_z^2\right), \qquad \Delta\phi = 2\left(u_r\phi_r + u_z\phi_z\right), \tag{27}$$

which determine ϕ and f. Eqs. (24,25) become

$$\frac{k_r}{r} = u_r^2 - u_z^2 - e^{-2u}\left(\phi_r^2 - \phi_z^2\right), \qquad \frac{k_z}{r} = 2u_r u_z - 2e^{-2u}\phi_r\phi_z \tag{28}$$

and determine k by integration, while Eq.(26) holds identically and is redundant. When $\phi = 0$, Eq. (27) becomes the Laplace equation for u and Eq. (28) gives $k(u)$. These are the axially-symmetric vacuum equations, also discovered by Weyl.

Now let us make the assumption that the gravitational and the electric potential have the same equipotential surfaces, $f = f(\phi)$. Eq. (27) yields

$$\left(f_{\phi\phi} - 2\right)\left(\phi_r^2 + \phi_z^2\right) = 0 \tag{29}$$

and that's how the quadratic relation (12) appears. Replacing it in Eq. (27) one comes to an equation for ϕ

$$\Delta\phi = \frac{B + 2\phi}{1 + B\phi + \phi^2}\left(\phi_r^2 + \phi_z^2\right). \tag{30}$$

We put for definiteness $B \geq 0$. Eqs. (12,30) with $B \leq 0$ are obtained by changing the sign of ϕ. The general solution of Eq. (30) is not known. However, let us make one more assumption, that ϕ depends on r, z through some function $\psi(r, z)$. Eq. (30) becomes

$$\frac{\phi_{\psi\psi}}{\phi_\psi} - \frac{(B + 2\phi)\,\phi_\psi}{1 + B\phi + \phi^2} = -\frac{\Delta\psi}{\psi_r^2 + \psi_z^2}. \tag{31}$$

If ψ satisfies the Laplace equation $\Delta\psi = 0$, $\phi(\psi, B)$ is determined implicitly [8] from

$$\psi = \int \frac{d\phi}{1 + B\phi + \phi^2}. \tag{32}$$

A very important equality follows

$$\phi_i = f\psi_i, \qquad \bar{\phi}_i = f(\phi)\,\bar{\psi}_i, \tag{33}$$

where $i = r, z$. Eq. (28) becomes

$$k_r = \frac{D}{4} r\left(\psi_r^2 - \psi_z^2\right), \qquad k_z = \frac{D}{2} r\psi_r\psi_z, \tag{34}$$

where $D = B^2 - 4$. Obviously, $k \sim \kappa$ always, making it much smaller than u, which is proportional to $\sqrt{\kappa}$.

Thus in Weyl electrovac solutions the harmonic master potential ψ determines the electric and the gravitational fields like u does this in the vacuum case. One may go further and find a relation between ψ and u, transforming Weyl electrovacs into Weyl vacuum solutions, although usual transformations work the other way round [3]. In particular solutions ϕ is usually proportional to the charge, $\phi = q\tilde{\phi}$. Eq. (32) shows that $\psi = q\tilde{\psi}$ where $\tilde{\psi}$ is harmonic and finite when the electric field is turned off by $q \to 0$. In this limit, when B does not depend on q, we have $f \to 1$ from Eq. (12) and $k \to 0$ from Eq. (34). Trivial flat spacetime is the result. However, if $B = \tilde{B}/q$ then $f \to 1 + \tilde{B}\tilde{\phi}$, $f_i \to \tilde{B}\tilde{\phi}_i$ and from Eq. (33) it follows that $u = \tilde{B}\tilde{\psi}/2$ and is harmonic. Eq. (34) then gives the vacuum expression for k. Hence, we obtain a Weyl vacuum solution (induced by some mass) with the same, up to a constant, harmonic function ψ. A mass term has appeared out of the vanishing charge.

Let us go back to the electrovac problem. One can add a constant ψ_0 to ψ in order to satisfy the conditions $\psi \to 0, f \to 1, \phi \to 0$ at infinity or when the electric field is turned off. The integral in Eq. (32) can be evaluated analytically and the dependence $\phi\left(\psi, B\right)$ made explicit. There are three cases, according to the sign of D. The simplest one is $D = 0$ ($B = 2$). Then f becomes a perfect square and $\psi_0 = -1$,

$$\phi = -1 - \frac{1}{\psi + \psi_0} = \frac{\psi}{1 - \psi}, \qquad f = \left(1 - \psi\right)^{-2}. \tag{35}$$

When $D < 0$ Eq. (34) gives $-D < 4$ and trigonometric functions appear

$$\phi = -\frac{B}{2} + \frac{\sqrt{-D}}{2} \tan \frac{\sqrt{-D}}{2} \left(\psi + \psi_0\right), \qquad \psi_0 = \frac{2}{\sqrt{-D}} \arctan \frac{B}{\sqrt{-D}}, \tag{36}$$

$$f = -\frac{D}{4\cos^2 \frac{\sqrt{-D}}{2} \left(\psi + \psi_0\right)} = \left(\cos \frac{\sqrt{-D}}{2}\psi - \frac{B}{\sqrt{-D}} \sin \frac{\sqrt{-D}}{2}\psi\right)^{-2}. \tag{37}$$

When $\psi_0 \equiv 0$ these formulas coincide with the Bonnor's ones [13].

Finally, when $D > 0$ there exist two expressions for the integral, one as a logarithm, the other in hyperbolic functions. They lead to

$$\phi = -\frac{B}{2} - \frac{\sqrt{D}}{2} \coth \frac{\sqrt{D}}{2} \left(\psi + \psi_0\right) = \frac{2\left(e^{\sqrt{D}\psi} - 1\right)}{B + \sqrt{D} - \left(B - \sqrt{D}\right) e^{\sqrt{D}\psi}}, \tag{38}$$

$$f = \frac{D}{4\sinh^2 \frac{\sqrt{D}}{2} \left(\psi + \psi_0\right)} = \left(\cosh \frac{\sqrt{D}}{2}\psi - \frac{B}{\sqrt{D}} \sinh \frac{\sqrt{D}}{2}\psi\right)^{-2}, \tag{39}$$

$$e^{\sqrt{D}\psi_0} = \frac{B - \sqrt{D}}{B + \sqrt{D}}. \tag{40}$$

According to Bonnor (who does not introduce ψ_0) the expressions for f and ϕ in the case $D > 0$ are obtained from Eq. (36) by continuation of $\sqrt{-D}$ to imaginary values and consequently tan $\to$ tanh, cos $\to$ cosh. This, however, holds when $\left(2\phi + B\right)^2 < D$, i.e., $4f < 0$, which is unphysical. In the physical case we must also do the replacement tanh $\to$ coth, cosh $\to$ sinh. The above discussion shows that the point $B = 2$ has a privileged position, unlike the point $B = 0$.

The Weyl solutions were derived with two assumptions imposed on the system: $f = f\left(\phi\right)$; $\phi = \phi\left(\psi\right)$, $\Delta\psi = 0$. They are particular solutions of Eqs. (27,28). However, when the symmetry is stronger than axial and the fields depend on just one coordinate x (not necessarily cylindrical, but a function of r, z), they comprise the general solution. For then $f\left(x\right)$ and $\phi\left(x\right)$ obviously are functionally related and one can always find $X\left(x\right)$ so that $\Delta X\left(x\right) = 0$. Taking $\psi = X\left(x\right)$ and expressing f and ϕ as functions of X, leads inevitably to the Weyl solutions. In the case of plane symmetry $x = z$, $X = x$. Cylindrical symmetry gives $x = r$, $X = \ln x$. Spherical symmetry has $x = \sqrt{r^2 + z^2}$, $X = 1/x$.

We have described the advantages of WMP solutions in inducing a powerful gravitational force. Some natural questions appear: is it possible to create such fields in a laboratory? What kind of charged sources should we take? Point [14] and line [15] sources lead to singularities and other problems. Therefore, we take a charged, closed, rotationally-symmetric surface with invariant density of the surface charge σ_s. The electrostatic theorem of Gauss has a generalization in general relativity [7, 16]. Integrating the r.h.s. of Eq. (5) one obtains the total charge contained in some volume

$$e=\frac{1}{c}\int J^{0}\sqrt{-g}d^{3}S=\int\sigma_{3}d^{3}S,\quad \sigma_{3}=\sigma\sqrt{-g^{(3)}},\tag{41}$$

where σ_3 is the three-dimensional invariant density and $g^{(3)}$ is the determinant of the space part of the metric. When the charge is attached to a surface, one should use the surface charge density σ_s instead. Integration of the l.h.s. of Eq. (5) leads to a relation between e and the electric flux through a closed surface S, encompassing the charged volume

$$4\pi e=\int_{S}\left[F^{01}\frac{\partial\left(x_{2},x_{3}\right)}{\partial\left(u,v\right)}+F^{02}\frac{\partial\left(x_{3},x_{1}\right)}{\partial\left(u,v\right)}+F^{03}\frac{\partial\left(x_{1},x_{2}\right)}{\partial\left(u,v\right)}\right]\sqrt{-g}dudv.\tag{42}$$

For Weyl solutions

$$F^{0i}\sqrt{-g}=-r\bar{\psi}_{i},\tag{43}$$

which is the result for flat spacetime. Hence, Eq. (42) becomes the Gauss theorem in classical electrostatics, but with ϕ replaced by ψ, which satisfies the Laplace equation. This fact was already stressed by Bonnor [4, 13] but in view of its importance we have discussed it again. Following a well-known procedure, we obtain a boundary condition on S for the jump of the normal component $\bar{\psi}_n$:

$$-\bar{\psi}_{n}|_{-}^{+}=4\pi\bar{\sigma}_{s}.\tag{44}$$

When ψ is given on S, there are two well-defined Dirichlet boundary problems and it may be continued as a harmonic function inside and outside S. If $\alpha\leq\psi_s\leq\beta$, these inequalities hold for ψ throughout space and it will be regular. Then f, k and ϕ are found from ψ in a manner already explained. The jump of ψ_n at S determines the source σ_s. The inverse is also true. When σ_s is given, there is a unique global ψ, satisfying Eq. (44). Consequently, for any distribution of charges on S one can find the electric and gravitational fields they induce. The same can be done when S is infinite and (or) not closed, but singularities may creep into the solutions.

Finally, by replacing Eq. (33) into Eq. (13) and expressing $f=f\left(\psi\right)$ we obtain

$$g_{i}=\frac{1}{2}\sqrt{G\left(D+4f\right)}\bar{\psi}_{i}=\sqrt{Gf}\bar{\psi}_{i}|_{B=2}.\tag{45}$$

For realistic EM-fields f is very close to one and this Maxwellian effect is the only one to be observed. Typical Einsteinian effects like light bending, gravitational redshift, time delay, changes in lengths are not enhanced by c^2 and are negligible.

Some questions immediately arise. Why do we get a Weyl solution for an arbitrary σ_s when Weyl fields are not the most general solutions of Eq. (27)? Why ϕ,f,k depend not alone on ψ but also on the constant B? How is its value determined, can we increase it, to enhance the effect of root gravity? In order to answer them we must return to the starting point, Eq.(3). Even when the EM-field is absent, Eq. (3) still has a number of non-trivial vacuum solutions, including e.g. gravitational waves. The situation is similar to classical electrodynamics without sources. Non-trivial solutions exist, but have to be time-dependent. These are the well-known electromagnetic waves. General relativity is a highly non-linear theory and vacuum solutions exist also in the static case. Their sources are well-hidden masses and even today there is a gap between the mathematical derivation of solutions [3] and their physical interpretation [17]. When EM-fields are turned on, these parasitic masses do not disappear and obscure the pure effect of electromagnetism on gravity. Let us try to get rid of them, step by step. First of all, the metric should inherit the symmetry of EM-fields. Let us confine again ourselves to axial symmetry. There are a lot of generation techniques, which produce non-Weyl solutions of Eq. (31). Most of the methods (see Ref. [3], Ch.34) start from the reformulation of Eq. (27) in terms of the Ernst potential $E=f-\phi^2$ [18], which is real in the absence of rotation,

$$f\Delta E=\nabla f\nabla E,\quad f\Delta\phi=\nabla f\nabla\phi.\tag{46}$$

The general solution of the Ernst equations can be found when the behaviour of $E\left(z\right)$ and $f\left(z\right)$ on the axis is given. It determines the multipole structure and is useful in astrophysics for modelling the gravitational field of stars. The presence of masses is welcomed, since they give the most substantial gravitational effect, followed by rotation (it can be incorporated into the formalism), magnetic fields and electric charge at the last place. The Ernst equation is much more difficult than the Laplace one and Dirichlet boundary value problems for it were discussed only recently [19]. On the other side, a harmonic function may be easily restored from its values on the axis $\psi(z)$ [20, 21]

$$\psi(r,z)=\frac{1}{\pi}\int_0^\pi \psi(z+ir\cos\theta)\,d\theta. \tag{47}$$

This real expression was used in general relativity for axially-symmetric static vacuum solutions [22], but it is not difficult to adapt it to Weyl electrovacs too. In the general solution $f(z)$ is not correlated with $\phi(z)$ and can be arbitrary, even when $\phi(z)$ vanishes, signalling the presence of masses. In the Weyl solution the metric goes to flat Minkowski spacetime when $\psi(z)$ vanishes. Thus the general solution expands over the Weyl one by the addition of masses. Probably the same is true for the solutions of Eq. (30), which also contain root gravity terms, since Eq. (12) is satisfied. The reason is that Weyl solutions form a complete system, covering the effect of any charge distribution and more general solutions can include the only other source of gravitation. More precisely, Weyl fields form an overcomplete system due to B which is not fixed by σ_s. In the case of plane, spherical, spheroidal or cylindrical symmetry they comprise the most general solution of the Ernst equation (46). It is quite improbable that the unwanted masses should disappear exactly in these cases, so the only way to show their presence is through the value of B.

A logical step is to accept that ψ plays the role of ϕ in any situation in electrogravity, not only for charged surfaces. Hence, when the gravitation created by charges is taken into account, it seems that Weyl fields generalize the solution to classical electrostatic problems. The physical electric field is found with the help of Eq. (33)

$$E_{(i)}=-\left(g_{00}g_{ii}\right)^{-1/2}\bar{\phi}_i=-fe^{-k}\bar{\psi}_i. \tag{48}$$

Now, since f,k are extremely close to one and zero respectively, one can do a perturbation theory around the exact Weyl solutions and set $E_{(i)}\approx-\bar{\psi}_i$. In the present case WMP fields are similar to instantons, monopoles, solitons and other non-perturbative exact solutions in quantum field theory. With high precision all electrostatic formulae hold also in the Einstein-Maxwell theory. The only new effect is the appearance of an electromagnetically induced gravitational acceleration, which reads from Eq. (45), again with high precision,

$$g_i=-\frac{B}{2}\sqrt{G}E_i. \tag{49}$$

We shall give arguments in the following that $B=2$, when unbiased by parasitic masses. Therefore, as already explained, measurable g_i are present from the available today electric and magnetic fields. One can reach g_e when $E_i=1.14\times10^9V/cm=3.8\times10^6CGS$ or $H_i=380T=3.8\times10^6G$. The lines of acceleration follow the electric field lines. Test particles will stay in equilibrium if they are charged and the relation between their mass m and charge e is

$$|e|=\sqrt{G}m. \tag{50}$$

Root gravity has some peculiar features. Changing the direction of E_i one changes the direction of g_i and when it points upwards with respect to the Earth's surface one has "anti-gravity". This is true because in our perturbation theory accelerations from electric fields and masses like the Earth or laboratory masses are added as usual vectors. The exact Weyl solution is necessary to clarify the gravitational induction in a laboratory set-up of finite size in space, where E_i is present. Although we have used a long-range interaction to induce another long-range interaction, in reality static EM-fields are always confined. Eq. (49) shows that putting a Faraday cage on E_i confines g_i too. It is understandable that when non-mass sources of gravitation are applied, the appearance of gravitational monopole terms (usually considered as mass terms) is not obligatory. Their existence is usually based on the Whittaker's theorem [16], which demonstrates the influence of some combination of the $T_{\mu\nu}$ components (called gravitational mass) upon the gravitational acceleration. The relation is given in terms of surface and volume integrals, appearing when the time-time component of the Einstein equations (1) is integrated. In this way it is not something separate and additional to them, but a consequence that can't contradict the conclusions following from them. Concretely, for Weyl fields this theorem is just the Gauss theorem for *grad u*. In the case of electromagnetic sources the "gravitational mass" is in fact some kind of "energy", inducing gravitational acceleration not necessarily with a monopole term. We avoid arguments based on the energy of the gravitational field because its density is not a tensor and there are at least five energy-momentum complexes [23], each with its own merits. Of course, some small mass term in the acceleration will always exist, due to the mass of the surface S. However, it will be of usual perturbative nature, many orders of magnitude smaller than root gravity.

Let us turn now to the case of magnetostatics. As was mentioned before, the analogue of ϕ is λ. It should be replaced in Eqs. (12,13,27,33,35,36,38). The analogue of Eq. (42) vanishes because there are no magnetic charges. One should take a closed surface with a surface current. In an axially-symmetric problem it has just one component, J_φ. A Weyl magnetostatic solution was given for the first time by Papapetrou in 1947 [6]. The analogy with electrostatics was investigated by Bonnor [4, 13] who showed that ψ is equivalent to the scalar magnetic potential. Then Eq. (44) should give the jump of the tangential to the surface component H_t which follows classically from $\bar{\psi}$

$$H_t|_{-}^{+} = \frac{4\pi}{c} J_\varphi \tag{51}$$

and is perpendicular to J_φ. Eqs. (48,49) still hold with $E_{(i)} \rightarrow H_{(i)}$, so that the gravitational effects of static magnetic fields mirror those in electrostatics. One can also introduce a vector potential, corresponding to ψ, which is more convenient in classical magnetostatics. However, in order to find the metric, the scalar potential ψ for each classical problem should be calculated too. The fields mainly of linear sources, like a current loop [15], and disks [24, 25] have been examined without discussing the presence of root gravity.

IV. CONNECTIONS AND GENERALIZATIONS

In the general stationary case the interval reads

$$ds^2 = f\left(dx^0 + \omega_a dx^a\right)^2 - f^{-1}\gamma_{ab} dx^a dx^b, \tag{52}$$

where ω_a is the gravitomagnetic potential ($a = 1, 2, 3$) and γ_{ab} is the three-dimensional metric. In the general static case $\omega_a = 0$ and in its electrostatic subcase the Einstein-Maxwell equations read

$$\Delta u = e^{-2u} \nabla\phi\nabla\phi, \quad \Delta\phi = 2\nabla u \nabla\phi, \tag{53}$$

$$R_{ab}^{(3)} = 2u_a u_b - 2e^{-2u}\phi_a\phi_b, \tag{54}$$

where $F_{0a} = -\bar{\phi}_a$. In magnetostatics $\phi \rightarrow \lambda$, the latter being defined by

$$F^{ab} = (-g)^{-1/2}\, \varepsilon^{abc}\bar{\lambda}_c. \tag{55}$$

This equation generalizes Eq. (18). The gradients, the Laplacian and the three-dimensional Ricci tensor are with respect to the metric γ_{ab}. Eqs. (53,54) generalize Eqs. (27,28). However, γ_{ab} also enters Eq. (53), making all equations interconnected and the system very difficult to deal with. In the special case when Eq. (12) holds with $B = 2$, γ_{ab} becomes flat and Eq. (53) can be solved, since it decouples from Eq. (54). The result is Eq. (35) with a harmonic $\psi(\varphi, r, z)$. Usually one takes $1 - \psi = U$ and $\phi = U^{-1}$, obtaining the already mentioned Majumdar-Papapetrou solutions from 1947 [5, 6], which are conformastatic. Some years later Ehlers [26, 27] gave transformations to derive such fields from vacuum ones. Similar transformations were given by Bonnor [28] and with the help of the TWS method [29, 30]. The latter was applied to the RN solution [30, 31].

When there is no space symmetry present, the simplest harmonic function in cartesian coordinates is

$$U(x, y, z) = 1 + \sum_i \frac{Gm_i}{c^2 r_i}, \quad r_i = \left[(x - x_i)^2 + (y - y_i)^2 + (z - z_i)^2\right]^{1/2}. \tag{56}$$

It can be shown that the sources are point monopoles with masses m_i and charges e_i [7, 14], connected by Eq. (50), which is true also in the Newtonian theory. It ensures the equilibrium between the electric and gravitational forces among the sources. Such multi-black-hole solutions satisfy certain uniqueness theorems and possess axially-symmetric reduction with the points staying on the z-axis [32]. They have been generalized to expanding cosmological solutions [33, 34].

Charged dust solutions with mass density μ and charge density σ should be mentioned too. For this purpose one must add to the r.h.s. of Eq. (3) $T^{\mu\nu} = \mu c^2 u^\mu u^\nu$ and introduce the current in Eq. (5). In the general static case one can impose the Newtonian equilibrium condition

$$\pm\,\sigma = \sqrt{G}\mu, \tag{57}$$

which is the density analogue of Eq. (50), but here it holds for the sources of the field, not for a test-particle. Then one gets [35]

$$f = (C + \phi)^2, \tag{58}$$

where C is some constant. Like in the multi-black-hole case one puts $\phi = U^{-1}$, $f = U^{-2}$, but U is not harmonic; it satisfies the non-linear equation [5, 35]

$$\Delta U = \pm \frac{4\pi\sqrt{G}}{c^2}\sigma U^3. \tag{59}$$

Charged dust clouds of spherical or spheroidal shape and obeying Eq. (59) have been studied extensively in an astrophysical context [36, 37]. They have a number of interesting properties when compared to usual stars: their mass and radius may be arbitrary, very large redshifts are attainable, their exteriors can be made arbitrarily near to the exterior of extreme charged black holes. In the spherical case the average density can be arbitrarily large, while for any given mass the surface area can be arbitrarily small. When their radius shrinks to zero, many of their characteristics remain finite and non-trivial.

Some more general solutions of Eq. (59) have been given too [38, 39]. They include the case of constant μ with U given in terms of a Jacobi elliptic function. The idea that μ may be concentrated on surfaces was also discussed [40]. Thin dust shells of spherical, cylindrical or plane shape were given as examples. Finally, a magnetostatic dust solution with functional dependence between f and A_φ is also known [41].

The charge density required to satisfy Eq. (57) is quite small. It is sufficient that in a sphere of neutral hydrogen one atom in about 10^{18} had lost its electron [36]. However, all charged dust models have one essential flaw: the equilibrium is very delicate and unstable and a slight change in σ would cause the cloud to expand or contract. On the other hand, the models discussed in the present paper do not depend on equilibrium conditions. They require just the creation of strong enough electric or magnetic fields. Of course, if the charged cloud is divided into parts with positive and negative particles (or ions), this can also induce EM-fields and root gravity.

The appearance of the WMP relation (12) was studied also when the dust is pressurized, i.e., in the case of charged perfect fluids and a gravitational model for the electron was put forth [42]. In both cases many different relations between f and ϕ are possible. Spherical charged perfect fluid interior solutions are reviewed in Ref. [43].

V. ELECTROSTATIC EXAMPLES

In this section two simple experimental set-ups are given where the effects of root gravity may be detected. They involve plane, spherical, and spheroidal symmetry. In these cases Weyl fields represent the most general electrovac solution and there is no ambiguity as to their appearance.

A. Plane symmetry

The study of plane-symmetric electrovac metrics goes back to 1926 [44], but for a long time their Weyl nature remained unrecognized. A plane-symmetric example of a WMP field was given first by Papapetrou [6]. Later Bonnor studied fields with $\phi = \phi(z)$, which include also some non-Weyl solutions [13]. Let us discuss the gravitational field of a uniformly charged plane with charge density σ. The classic Maxwell potential (which becomes the master-potential) has both plane and mirror symmetry

$$\psi = -2\pi\sigma\,|z| \equiv q\,|z|\,. \tag{60}$$

It vanishes at the plane $z = 0$ and goes to infinity when $|z| \to \infty$. In the case $D = 0$ we should replace Eq. (60) into Eq. (35). Kar has proposed a coordinate system where ϕ becomes harmonic, $\phi = q\,|z'|$. It gives a constant electric field and further deepens the analogy with classical electrostatics. Then the interval becomes

$$ds^2 = f\left(dx^0\right)^2 - f^{-1}\left(dr^2 + r^2 d\varphi^2\right) - f^{-3}\left(dz'\right)^2, \tag{61}$$

where $f = \left(1 + q\,|z'|\right)^2$. When $D \neq 0$ one sees from Eq. (34) that k depends on r and in fact

$$k = -\frac{D}{8}q^2 r^2. \tag{62}$$

The metric does not inherit the symmetry of its source. Now the Kar's gauge gives $\phi(z) = -B/2 + q\,|z'| + \alpha$. Requiring flatness at $z' = 0$ [13] one obtains $\alpha = B/2$ and Eqs. (36,37) yield

$$ds^2 = f\left(dx^0\right)^2 - e^{q^2\left(1-\alpha^2\right)r^2}\left(f^{-1}dr^2 + f^{-3}dz'^2\right) - f^{-1}r^2 d\varphi^2, \tag{63}$$

where $f = 1 + 2q\alpha\left|z'\right| + q^2 z'^2$. The non-inheritance is obvious again. The $D = 0$ case is obtained when $\alpha = \pm 1$ and is the only one with plane symmetry. This is a strong argument that $B = 2$. Other values of B (including $B = 0$) introduce, in addition, the parameter α, which does not originate from the electric field, unlike the charge parameter q. Probably it is invoked by hidden mass sources. As for the condition $f\left(z = 0\right) = 1$, its necessity will become clear in the following. In conclusion, the only plane-symmetric Weyl metric is given by Eq. (61). Another example of non-inheritance was given in Ref. [45], where the metric is not rotationally invariant.

In order to obtain a regular global solution, one must satisfy the junction conditions [46] at the plane. The metric is continuous there. The extrinsic curvature reads

$$K_{aa} = \frac{1}{2}\left|g_{zz}\right|^{-1/2}\left(g_{aa}\right)_z, \tag{64}$$

where $a = x^0, \varphi, r$ and there is no summation. Its eventual jump at the plane determines the required energy-momentum tensor S_{ab} of the massive surface layer

$$\kappa S^a_{\ b} = \gamma^a_{\ b} - \delta^a_b\gamma, \quad \gamma^a_{\ a} = K^a_{\ a}|^+_-, \quad \gamma = \sum_a \gamma^a_{\ a}. \tag{65}$$

For the Weyl form of the axially-symmetric metric one has

$$\kappa S^0_{\ 0} = e^{u-k}\left(2u_z - k_z\right)|^+_-, \tag{66}$$

$$\kappa S^\varphi_{\ \varphi} = -e^{u-k}k_z|^+_-, \quad \kappa S^r_{\ r} = 0. \tag{67}$$

In our case $k_z = 0$ and even $k = 0$. At first sight, the jump in the acceleration u_z requires the introduction of mass on the charged plane. However, its true cause is the electric field present in the space around the plane and induced by the charge distribution on it. This is seen when one makes use of Eqs. (12,33)

$$u_z = \left(1 + \phi\right)\psi_z. \tag{68}$$

Obviously, the jump in u_z at $z = 0$ is due to the jump in the master-potential ψ_z and consequently to the presence of charge. The charged surface layer of the plane causes the jump in the extrinsic curvature too. For Weyl fields the usual procedure of introducing mass and pressures on the surface represents an attempt to model the influence of the electric field upon the metric by traditional mass sources [24, 25, 47]. When ψ from Eq. (60) is replaced in g_i, the plane will be attractive for one sign of q, as if there was positive mass on it, but for the other sign of q it will be repulsive as if it were made of exotic negative mass, which breaks the energy conditions. There is no paradox because the true creator of these effects is the energy-momentum tensor of the electric field, which always satisfies all three energy conditions.

Now let us put a second plane at $z = d$, charged in the opposite way. The electric field is confined between the two planes and ψ reads

$$\psi = \frac{\psi_2}{d}z, \quad E_z = -f\bar{\psi}_z. \tag{69}$$

Here ψ_2 is the potential of the second plane ($\psi_1 = 0$). It is related to the charge density by

$$\psi_2 = 2\pi\sigma\frac{d}{\varepsilon}, \tag{70}$$

where ε is the dielectric constant. Up to now we have considered the case $\varepsilon = 1$. More generally, ε enters Eq. (3) because the energy $T_{00} \sim \varepsilon E_i^2$ and consequently $T_{\mu\nu} \to \varepsilon T_{\mu\nu}$. When ϕ absorbs the constants in Eq. (10) it will pick also $\sqrt{\varepsilon}$. The same is true for ψ. The acceleration formula (45) becomes in a dielectric medium

$$g_z = \sqrt{G\varepsilon f}\frac{\bar{\psi}_2}{d} \approx 2.58 \times 10^{-4}\frac{\sqrt{\varepsilon}}{d}\bar{\psi}_2. \tag{71}$$

Taking finite disks instead of planes one obtains the usual capacitor. The field around its centre will be plane-symmetric, while at the rim it will depend on r too. This may be diminished by careful electric shielding of the capacitor. Outside there will be a vacuum flat metric, which joins smoothly the interior due to the conditions

$f = const$ on the plates. One comes to the conclusion that the capacitor will be subjected to practically constant gravitational force F_g in the z-direction,

$$F_g = \sqrt{G\varepsilon}\frac{M}{d}\bar{\psi}_2 = \sqrt{G\varepsilon}\mu S\bar{\psi}_2, \tag{72}$$

where M is the mass of the dielectric, μ is its mass density and S is the area of the plate. This force is very different from the electric force, trying to bring the plates together

$$F_E = \frac{\varepsilon^2 S\bar{\psi}_2^2}{2\pi d^2}. \tag{73}$$

The latter is neutralized by the mechanical construction of the capacitor. If it is hanging freely, the effect of F_g may be tested experimentally. To increase the acceleration it is advantageous to make d small (typically $0.1cm \leq d \leq 1cm$), to raise the potential difference ψ_2 between the plates up to $2 \times 10^4 CGS$ and to take a material with high ε. Some examples are gases ($\varepsilon \approx 1$), quartz (4.5), glycerine (56.2), water, electric ceramics (81), rutile (TiO_2) with $\varepsilon = 170$. Ferroelectrics are even better if one can cope with their hysteresis and tendency for saturation in strong fields. Barium titanate ($BaTiO_3$) and many others have ε in the range of 10^4. Thus $\sqrt{\varepsilon}/d$ may reach in principle 10^3 and the maximum acceleration $g_{z,\max} = 5.2 g_e$ is more than enough to counter Earth's gravity. In a more modest attempt one can take $\bar{\psi}_2 = 100kV$ and $\sqrt{\varepsilon}/d = 10^2$ to get about one percent of g_e. Curiously, the same factors $\varepsilon, d, \bar{\psi}_2$ are even more important in F_E, which is typically much bigger than F_g.

As we shall see, root gravity effect is strongest in this first capacitor example and, more generally, when the electric (magnetic) field lines are parallel. The gravitational acceleration is not strictly constant, having an unobservable dependence on z. One cannot obtain root gravity terms by studying fields with constant acceleration [48]. Earth's field also shares these "shortcomings" of artificial gravity: the acceleration's directions are not parallel but meet at the planet's centre and its magnitude decreases with height.

In the traditional approach to the charged plane its metric depends directly on t and z, bypassing the axially-symmetric step, see [3], p234. The Weyl nature is then completely obscured. Some plane metrics are induced by null EM-fields, like the special pp-wave given by Eq. (15.18) from [3] or the Robinson-Trautman solution, given by Eq. (28.43) from the same reference. They are non-static because null EM-fields are incompatible with static metrics, [3], Theorem 18.4, [49]. We also do not discuss the non-null homogenous and uniquely conformally flat Bertotti-Robinson solution, [3] Sec.12.3. All other plane-symmetric solutions with non-null EM fields have been found by Letelier and Tabensky [50]. The metric is either static or spatially homogenous. We are interested in the static branch. It has been reviewed in Ref. [51], however, the electric field was not discussed and the ties with the Weyl fields remained unelucidated. Let us clarify this issue now.

The metric reads in cylindrical coordinates

$$ds^2 = Kc^2 dT^2 - N\left(dR^2 + R^2 d\Phi^2\right) - PdZ^2, \tag{74}$$

where K, N, P depend on Z and there is one relation between them. It allows to express K and P as functions of N

$$K = \frac{1}{N}\left(\beta_1 + \beta_2\sqrt{N}\right), \quad P = \frac{N_z^2}{4\mu^2}\left(\beta_1 + \beta_2\sqrt{N}\right)^{-1}, \tag{75}$$

$$\beta_1 = \frac{\eta^2}{4\mu^2}, \quad \beta_2 = 1 - \beta_1, \tag{76}$$

where η and μ are constants related to the charge and mass density on the plane. For the electric field we get from Eq. (5)

$$E_z = -\phi_z = -\frac{\eta\sqrt{KP}}{2N}, \quad \phi = \frac{\eta}{2\mu}\left(\frac{1}{\sqrt{N}} - 1\right). \tag{77}$$

We have chosen the boundary conditions $\phi(0) = 0$, $N(0) = 1$. It is easily seen that

$$K = 1 + B_0\phi + \phi^2, \quad N = \left(1 + \frac{2\mu}{\eta}\phi\right)^{-2}, \tag{78}$$

$$B_0 = \frac{2\mu}{\eta} + \frac{\eta}{2\mu} = \frac{2 - \beta_2}{\sqrt{1 - \beta_2}}. \tag{79}$$

Thus K, the analogue of f, obeys Eq. (12) and root gravity terms are present. When ϕ is turned off by $\eta \to 0$ ($\beta_1 \to 0$), B_0 does not stay constant. It increases to infinity instead, so that $B_0\phi$ remains finite. This triggers the "mass out of charge" mechanism described in Sec.III and we end with the vacuum solution for a massive plane with $K = N^{-1/2}$ [52]. Hence, the mass is not entirely of electromagnetic origin and the parameter μ is independent from η in general. Another limit is $\beta_2 \to 0$. This was explored first by McVittie [53]. Then $\beta_1 = 1$, $\eta = \pm 2\mu$, $B_0 = \pm 2$, $K = 1/N$.

One can absorb N in Eq. (74) by the change $Z \to N$. However, it is better to determine N from the relation between K, P and N which specifies the coordinate system. The conformastatic spacetimes studied by Weyl [8], Papapetrou [6] and Bonnor [13] have $N = P$. Kar [44] used two gauges, $KP = 1$ and $KP = N^2$ in his pioneering work. The second gives constant E_z, as seen from Eq. (77). McVittie [53] worked in the gauge $KP = N$. Patnaick [54] attacked the problem with $K = P$, which is the Taub's gauge in the vacuum case [55]. The same gauge was utilized by Letelier and Tabensky [50]. Unfortunately, the equations cannot be integrated explicitly in this gauge except for the McVittie limit.

In order to make comparison between the axially-symmetric approach and the traditional plane-symmetric approach one should use the gauge $N = P$. This differential equation for N is easily solved and yields

$$N = P = \left(1 - \mu Z + \frac{\beta_2\mu^2}{4}Z^2\right)^2, \quad K = \frac{1}{N}\left(1 - \frac{\beta_2\mu}{2}Z\right)^2, \tag{80}$$

$$\phi_z = \frac{\eta}{2N}\left(1 - \frac{\beta_2\mu}{2}Z\right), \quad \phi = \frac{\eta}{2\mu}\left[\left(1 - \mu Z + \frac{\beta_2\mu^2}{4}Z^2\right)^{-1/2} - 1\right]. \tag{81}$$

These expressions coincide with the $B = 2$ case, Eqs. (35,60) only in the McVittie limit where $\beta_2 = 0$ and consequently $q = \mu = \eta/2$, $\eta = -4\pi\sigma$. In this limit the fields in Kar's gauge [44, 51] coincide with Eq. (63) for $\alpha = 1$ after the identification $Z = z'$. The original McVittie's metric is obtained by passing to $e^{qz''} = qz' + 1$

$$ds^2 = e^{2qz''}\left(dx^0\right)^2 - e^{-2qz''}\left(dx^2 + dy^2\right) - e^{-4qz''}\left(dz''\right)^2. \tag{82}$$

In conclusion, the purely electric effect upon the metric of a charged plane is given by the McVittie solution, while the deviation of β_2 from zero (μ from $\eta/2$) signals the presence of mass in addition to the charge. In this way one can increase (very inefficiently) B_0 and the root gravity acceleration.

It can be seen [47] that in the vacuum case a mass surface layer has to be introduced in order to explain the mass term, appearing in the metric. However, the electrostatic field creates itself a monopole term, indistinguishable from the mass term (electromagnetic mass) and, hence, a jump in the extrinsic curvature. Therefore, it is not necessary to introduce compensating masses and pressures on the charged plane, in confirmation of what was established above.

B. Spherical symmetry

Let us study now the gravitational effect of electric fields with spherical symmetry. The master potential is

$$\psi = \frac{q}{\rho}, \quad \rho^2 = r^2 + z^2. \tag{83}$$

This is the exterior solution for a metal conductive sphere of radius ρ_1, charged to a potential $\psi_1 = q/\rho_1$. Inside the sphere $\psi = 0$ and the spacetime is flat. Outside we obtain the charged [56, 57] Curzon [58] solution. Eq. (34) gives

$$k = -\frac{Dq^2r^2}{8\rho^4}. \tag{84}$$

Like in the plane-symmetric case the metric does not inherit the spherical symmetry of ψ unless $D = 0$ ($B = 2$). This corresponds to the critically charged Curzon metric. Eq. (35) yields

$$\phi = \frac{q}{\rho - q}, \quad f = \left(1 - \frac{2q}{\rho} + \frac{q^2}{\rho^2}\right)^{-1}. \tag{85}$$

The charge in CGS units $\bar{q}$ is connected to q by $q = \frac{\sqrt{G}}{c^2}\bar{q}$. Eq. (45) gives for the acceleration

$$g_\rho = -\sqrt{Gf}\frac{\bar{\psi}_1\rho_1}{\rho^2}. \tag{86}$$

It has a maximum at the sphere and utilizing our maximum electric potential we obtain $|g_{\rho,\max}| = 5.06/\rho_1$. When $\rho_1 = 10cm$, one gets $0.5cm/s^2$. Formula (64) for the extrinsic curvature holds in the present case after the replacement $z \to \rho$ and $a = x^0$, θ, φ, i.e., the axially-symmetric element is written in spherical coordinates

$$ds^2 = e^{2u}\left(dx^0\right)^2 - e^{-2u}\left[e^{2k}\left(d\rho^2 + \rho^2 d\theta^2\right) + \rho^2 \sin^2\theta d\varphi^2\right]. \tag{87}$$

Expressions (66,67) for S_0^0 and S_φ^φ hold after the same change, while $S_\theta^\theta = 0$. Once again we argue that there is no mass surface layer, but only a charged one. As in the previous case, for one sign of q gravity becomes repulsive. The interaction between massive bodies with repulsive (negative mass) and attractive (positive mass) gravitation has been discussed in Refs [59, 60]. In our case repulsion is a natural property of the electric field and does not break the energy conditions.

Let us deform the charged sphere into an oblate or prolate spheroid. Its gravitational field is described best in the corresponding spheroidal coordinates x, y (to be distinguished from the cartesian coordinates in the previous sections)

$$r = \tau\left(x^2 \pm 1\right)^{1/2}\left(1 - y^2\right)^{1/2}, \quad z = \tau x y, \tag{88}$$

where τ is a parameter. In the prolate case one has

$$x = \frac{1}{2\tau}\left(l_+ + l_-\right) \equiv \frac{L}{\tau}, \quad y = \frac{1}{2\tau}\left(l_+ - l_-\right), \tag{89}$$

$$l_\pm = \sqrt{\left(z \pm \tau\right)^2 + r^2}. \tag{90}$$

The master potential is taken to depend only on x ($x = x_1$ is the surface of the charged spheroid). The harmonic solution is given by the Legendre function $Q_0(x)$

$$\psi = \frac{q}{2}\ln\frac{x-1}{x+1} \tag{91}$$

and generates the general solution of the Einstein-Maxwell equations for such symmetry. From Eq. (44) one has $q = 4\pi\sigma\left(x_1^2 - 1\right)$. This potential coincides in form with the Weyl rod for the vacuum γ-metric [17]. Eq. (34) gives

$$k = \frac{Dq^2}{8}\ln\frac{x^2-1}{x^2-y^2}, \tag{92}$$

which depends on y too, signalling non-inheritance, except when $B = 2$. The same conclusion follows in the oblate case, but there $\psi = -q\arctan 1/x$. Repulsive gravity also arises in these cases.

It is well-known that the unique spherically symmetric electrovac solution with mass M and charge $\bar{Q}$ is the Reissner-Nordström solution [61, 62]

$$ds^2 = \left(1 - \frac{2m}{R} + \frac{Q^2}{R^2}\right)c^2 dT^2 - \left(1 - \frac{2m}{R} + \frac{Q^2}{R^2}\right)^{-1} dR^2 - R^2\left(d\Theta^2 + \sin^2\Theta d\Phi^2\right), \tag{93}$$

$$\phi = \frac{Q}{R}, \quad m = \frac{GM}{c^2}, \quad Q^2 = \frac{G\bar{Q}^2}{c^4}. \tag{94}$$

Hence, at least the solution given by Eq. (85) should be transformable into the RN solution. Surprisingly, all three solutions described above are formally equivalent to its three cases; undercharged ($Q^2 < m^2$), critically (extremely) charged ($Q^2 = m^2$) and overcharged ($Q^2 > m^2$). The proof uses the Weyl form of the RN solution [32, 57]. The essential step is to apply Kar's gauge, in which ϕ becomes harmonic instead of ψ. When $D = 0$ we transform Eq. (85) by setting $R = \rho - q$, $\cos\Theta = z/\rho$ and obtain Eqs. (93,94) with $m = -q$, $Q = q$. The sphere $\rho = \rho_1$ transforms into the sphere $R = R_1 = \rho_1 - q$. When $D > 0$ we take Eq. (91) and fix q by the condition $Dq^2/4 = 1$. Then we utilize Eqs. (38,39) to find ϕ and f

$$\phi = -\frac{2}{\sqrt{D}\left(\frac{B}{\sqrt{D}} + x\right)}, \quad f = \frac{x^2-1}{\left(\frac{B}{\sqrt{D}} + x\right)^2}. \tag{95}$$

Making the identifications

$$\tau = \left(m^2 - Q^2\right)^{1/2}, \quad B = -\frac{2m}{Q}, \tag{96}$$

we obtain (together with Eqs. (88-92)) the formulas for the undercharged case from Ref. [32]. The transformation

$$R = \tau x + m, \quad \cos\Theta = y \tag{97}$$

maps this solution into the RN solution. The charged spheroid $x = x_1$ goes into the charged sphere $R = R_1 = \tau x_1 + m$. It is seen from Eq. (96) that $B \to \infty$ when $Q \to 0$ and $B\phi$ remains finite. The independent mass parameter m again arises from the "mass out of charge" mechanism. A similar chain of arguments connects the oblate spheroid case to the overcharged RN solution.

In conclusion, the electrically induced spherically symmetric gravitational field is given by the critically charged Curzon solution, which is equivalent to an extreme RN solution with $m = -Q$. There is a transformation of RN solutions with $m \neq \pm Q$ into spheroidal metrics with particular D. The distance between the spheroid's foci τ is related to the deviation of m from its electromagnetic value. Thus part of the exterior solution's mass is from electromagnetic origin. This has been long known for interior charged perfect fluid solutions [43]. The results are analogous to the plane-symmetric ones. The usual RN solution with mass and charge also exerts a repulsive force for certain values of its parameters [63, 64]. It appears, however, due to another mechanism. In the charged Curzon solution the sign of the charge is decisive and the region with repulsion occupies the whole exterior space. In the RN solution the mass is always positive, but enters the metric with a negative sign, so that a competition with the charge term is possible. The region with repulsion is finite, $R < Q^2/m$.

VI. MAGNETOSTATIC EXAMPLES

We have pointed out that magnetic fields produce the same effects as electric ones. One must replace ϕ by λ, E_i by H_i, while ψ becomes the magnetic scalar potential, determined by surface currents. This is confirmed also by the definition of the magnetic field [65]

$$H_i = -\frac{1}{2}\sqrt{-g}\varepsilon_{ikl}F^{kl}, \tag{98}$$

which gives, using Eq. (18)

$$H_i = -\bar{\lambda}_i = -f\bar{\psi}_i. \tag{99}$$

Eqs. (17,33) provide the connection between the scalar potential and the only component ($\bar{\chi} = A_\varphi$) of the vector potential

$$\bar{\chi}_z = r\bar{\psi}_r, \quad \bar{\chi}_r = -r\bar{\psi}_z. \tag{100}$$

It should be noted that in flat spacetime magnetostatics the formula $\vec{H} = curl\vec{a}$ involves the physical component in curvilinear coordinates $a_{(\varphi)} = A_\varphi/r$. In magnetogravity one has to find the scalar potential anyway, in order to obtain f.

In a general medium the energy is $T_{00} \sim \mu H^2$, where μ denotes now the magnetic constant and consequently $T_{\mu\nu} \to \mu T_{\mu\nu}$. Hence, ψ picks also the multiplier $\sqrt{\mu}$. Eq. (99) remains unchanged. Arguments, analogous to those for electric fields, lead to the conclusion that $B = 2$ for a pure magnetic effect upon gravity. Formula (45) for the acceleration becomes with high degree of precision

$$g_i = -\sqrt{G\mu}H_i. \tag{101}$$

It is similar to the electric case, Eqs. (49,71). The gravitational force is very different from the Lorentz force, acting upon charged particles

$$\vec{F}_L = e\vec{E} + \frac{e\mu}{c}\left(\vec{v} \times \vec{H}\right). \tag{102}$$

In it the electric force does not involve ε, while the magnetic does involve μ, but acts only on moving charges. There are terms, depending on the velocity, also in g_i. However, we are interested in the gravitational effect upon macroscopic bodies for which $v \ll c$, hence, we have neglected them and study only acceleration at rest.

The first magnetovac Weyl solution was given by Papapetrou [6]. Later Bonnor presented a number of examples [4, 15], including the gravitational field of a current loop. He found a very small effect, based on mass-energy considerations.

The magnetic field of a current loop is well-known and includes elliptic integrals. On the z-axis and at the centre the field simplifies

$$H_z\left(z,0\right)=0.2\pi I\frac{r_1^2}{\left(r_1^2+z^2\right)^{3/2}},\quad H_0\equiv H_z\left(0,0\right)=\frac{0.2\pi I}{r_1}, \tag{103}$$

where r_1 is the loop's radius in cm, H is measured in Gauss and the current I is measured in $Amps$. Setting $\mu=1$ we get from Eq. (103)

$$g_0\equiv\left|g_z\left(0,0\right)\right|=1.62\times10^{-4}\frac{I}{r_1}. \tag{104}$$

Earth's acceleration is reached when $H_e=3.8\times10^6G=380T$ or $I_e/r_1=6.05\times10^6A/cm$. For a laboratory set-up let us take $r_1=100cm$. If a lightning with cross-section of radius $r_0=10cm$ circles around the loop, one may take $I=10^5A$, the current density being $J=3.18\times10^2A/cm^2$ and $g_0=0.162cm/s^2$. One can reach g_e with a current of 6.05×10^8A.

It is advantageous to make the loop thicker, turning it into a finite solenoid. Let the inner radius be r_1, the outer radius be r_2 and the height be l. Then [66, 67]

$$H_0=F\left(\alpha,\beta\right)r_1J,\quad F\left(\alpha,\beta\right)=0.4\pi\beta\ln\frac{\alpha+\sqrt{\alpha^2+\beta^2}}{1+\sqrt{1+\beta^2}}, \tag{105}$$

where $\alpha=r_2/r_1$, $\beta=l/r_1$ and J is the current density. As an example, let us take $r_1=100cm$, $r_2=2r_1$, $l=2r_1$. Then $F\approx1$ and $J=H_0/100$. Now g_e is reached when $J_e=3.8\times10^4A/cm^2$.

One can increase the acceleration by creating magnetic fields in a ferromagnetic medium. Iron has $\mu_{\max}=5000$ ($\sqrt{\mu_{\max}}=70.7$). There are alloys, like supermalloy, which have $\mu_{\max}=8\times10^5$, $\sqrt{\mu_{\max}}=894.4$. Their saturation field is comparatively low, $H_s=8\times10^3G$ and the maximum is obtained roughly for one third of this value. The effective field in Eq. (101) will be $H_{eff}=\sqrt{\mu_{\max}}H_{\max}\approx238T$, which is of the order of H_e. Making a disc from this material with a current flowing in a strip around the rim, one gets the magnetic analogue of the moving capacitor example from Sec.V.A.

There are two issues which need clarification. The metric depends directly on the scalar potential, but ψ is multivalued in magnetostatics. If there is a current-carrying surface, the jump takes place there and probably the metric can be made continuous in this region of non-vanishing J. In the case of a current loop the jump can be arranged to take place on any surface, based on the loop. Symmetry considerations require to take the disk $z=0$, $\rho<r_1$ as such a surface. In spherical coordinates [15]

$$\psi=0.2\pi J\left[\pm1-\frac{\rho}{r_1}P_1+\frac{\rho^3}{2r_1^3}P_3-...+\left(-1\right)^{n+1}\frac{1.3...\left(2n-1\right)}{2.4...2n}\left(\frac{\rho}{r_1}\right)^{2n+1}P_{2n+1}+...\right], \tag{106}$$

where the sign coincides with $signz$ and $P_n\left(\cos\theta\right)$ are the Legendre polynomials. Now, let us remember that we have been unable to determine the sign of B and for definiteness worked with positive B. In fact, Eq. (35) should read

$$f=\left(1\pm\psi\right)^{-2}. \tag{107}$$

Using different signs for z positive or negative, and taking into account that $P_{2n+1}\left(\cos\frac{\pi}{2}\right)=0$, makes f continuous at $z=0$. The derivatives of ψ are single-valued, hence, k is also continuous. The acceleration will have a jump and change of sign at the disk within the loop, like it has on both sides of a charged plane (disk). We took the absolute value of z in Eq.(60), which plays the same role. This jump has been attributed to some mass surface layer [15], which in our view is not correct.

The second issue concerns the fact that the magnetic field far away from the loop has a dipole character and the acceleration does not contain a monopole term. As we have argued, this is not a tragedy, because electromagnetic fields can create artificial gravity in a confined region of space. Intuitively, this is more favourable energetically. Of course, the material of the loop always has some mass, which induces a monopole term, negligible with respect to the strong effect from root gravity.

VII. CONCLUSIONS

The results of this paper can be summarised as follows. Some of them are not new and the corresponding references are cited in these cases.

1) The gravitational acceleration at rest in Weyl-Majumdar-Papapetrou fields has a root gravity term, proportional to $c^2\sqrt{\kappa} = \sqrt{G}$, which is 10^{23} times bigger than the usual perturbative coefficient $c^2\kappa$. It is linear in the EM-fields, while the perturbative term is quadratic. Sizeable gravitational force exists (see Eq. (49)) although the metric is very close to the flat one (Maxwellian limit). Its explicit form determines up to a sign the important constant B. For its typical value $B = 2$ the Earth's acceleration is obtained in electric fields of order $10^9 V/cm$ and in magnetic fields of about $380T$. One can change the direction of g_i by changing the direction of E_i or H_i and confine g_i to a finite volume by confining the EM-fields.

2) The energy-momentum tensor (in particular, its energy component T_{00}) induces a change in the Ricci tensor $\sim \kappa$ according to the Einstein equations (3). This leads to changes in the metric and its acceleration, which can be $\sim \sqrt{\kappa}$ and may contain no monopole term. Then the formula $E = mc^2$ does not hold. Creating artificial gravity that is localized in space and has no long-distance mass terms seems energetically more favourable.

3) In axially-symmetric systems Weyl fields provide regular exterior and interior solutions to any distribution of charges or currents on a closed surface. They are determined by a master-potential ψ satisfying the Laplace equation. When the metric depends on just one coordinate (three commuting Killing vectors are present) the Weyl solution becomes the most general one. In truly axisymmetric cases it presumably determines the pure electromagnetic effect on gravity, while the solutions of the Ernst equation include hidden mass sources. The constant B (taken positive for definiteness) divides the Weyl solutions into three classes, according to $B = 2$, $B > 2$, $B < 2$. [13]. Among them $B = 2$ is privileged, being the simplest (conformastatic spacetimes). When $B \neq 2$, parasitic masses appear. In some cases the metric does not inherit the symmetry of the EM-source.

4) The gravitational force at rest induced by electric and magnetic fields is the same [9, 10, 11] unlike the Lorentz force, acting upon charged particles. The surface sources determine the master potential and not A_μ [4, 13].

5) There is a "mass out of charge" mechanism, which allows to obtain solutions with mass from the Weyl solutions. It clearly indicates the part of mass which is of electromagnetic origin. Eq. (12) still holds and root gravity term remains, but B is affected by the mass. Such solutions can incorporate the mass of the charged surface, which is always present in practice.

6) In the general static case a harmonic master potential for WMP fields appears only when $B = 2$ [5, 6]. Point or line sources have singularities, hence, one should use closed surface sources (shells). Another alternative is to use volumes of charged dust or perfect fluid, where the functional dependence $f(\phi)$ appears naturally as an equilibrium condition. For charged dust a harmonic potential can always be introduced, but the equilibrium is unstable [36, 37]. It is more realistic to charge conductive surfaces by applying potentials or pass currents through coils wound around them, relying on the motion of free electrons in metals. One can also create powerful fields by separating the positive and negative ions.

7) In order to construct global solutions around the charged shells, the fulfilment of the junction conditions is required. They have some subtleties in the Weyl case. Eq. (12) interrelates the Einstein and the Maxwell equations (27) and a jump in $T_{\mu\nu}$ results due to the jump in ψ, caused by the charges or the currents. A mass surface layer is not necessary. The acceleration is caused by the EM-field and not by often unrealistic fluid sources, trying to duplicate this effect.

8) The pure electric plane-symmetric effect on the metric is described by the McVittie solution, which is a Weyl solution of class $B = 2$. When $B \neq 2$ the Weyl metric is not plane-symmetric (non-inheritance of the source's symmetry). Solutions with mass and charge also contain a root gravity term.

9) A usual freely hanging capacitor may be used to test root gravity. Constant acceleration is induced in its dielectric, Eq. (71). Taking maximum potential difference, minimum distance between the plates and a material with very big dielectric constant (ferroelectric) one can obtain $g_z = 5.2 g_e$. This force is trying to move the capacitor in the z direction. It is smaller than the force of electric attraction between the plates.

10) The pure electric spherically-symmetric effect on the metric is described by the charged Curzon solution, Eq. (85), which is related by a simple transformation to the critically (extremely) charged RN solution. This is a Weyl solution with $B = 2$. When $B \neq 2$ non-inheritance of the source's symmetry results. The same is true for prolate and oblate spheroidal solutions. They are singular at the axis, so one can use them as exteriors to a charged spheroidal shell with Minkowski interior. Their gravity becomes repulsive in the whole outside region for one sign of the charge. Typically, the repulsive $g = 0.5 cm/s^2$. There is a formal coordinate transformation between RN fields with mass and charge and Weyl fields outside spheroidal charged shells.

11) The magnetic field of a current loop induces a gravitational force according to Eq. (104). The Earth's acceleration is reached when the current $I = 6 \times 10^8 A$. A thick loop (fat solenoid) requires for this goal current density of $3.8 \times 10^4 A/cm^2$. A ferromagnetic disc with a current strip around it is the magnetic analogue of the moving capacitor.

VIII. DISCUSSION

Root gravity has been overlooked in the past. One reason is the wide-spread use of relativistic units - already Weyl worked in them. The only papers on the subject (except the present one) written in non-relativistic units are by Ehlers [26, 27], but are extremely rarely cited.

Another reason is that the simpler solutions with plane, spherical, spheroidal or cylindrical symmetry were studied directly and were almost never treated as subcases of the axisymmetric solutions. This cut their ties to Weyl solutions from the very beginning. The RN solution was shown to belong to the Weyl class only in 1972 [32].

A third reason is the underestimation of exact solutions in favour of approximation schemes and numerical techniques. Root gravity escapes undetected by these methods. In fact, it sets a new scale $\sqrt{\kappa}$ or $\sqrt{G}$ and perturbations should be done around the exact Weyl solutions. WMP solutions have never been in the main trend of development of general relativity. The fact that the unique charged black hole is a WMP solution pushed their investigation towards multiple-charged-black holes [5, 6, 7, 14, 32, 33, 34], particle trajectories in these fields [68] or WMP-based wormholes [69]. A similar astrophysical issue are charged dust clouds (Bonnor stars) and their non-singular interpolation to black holes [36, 37, 70, 71]. Axisymmetric solutions have been studied mainly with the generation techniques for the Ernst equation. Papers on WMP solutions often appeared in local, old or unavailable journals.

A fourth reason concerns the motivation of the first researchers. Weyl himself was much more interested in his conformal theory of gravity, trying to unify gravitation and electromagnetism. He never returned back to the Weyl fields. McVittie presented his solution as an example to test one of the unified theories of Einstein. The latter concentrated his energy on the geometrical unification of the two fundamental long-range interactions.

In this paper we share another view, closer to the Rainich "already unified theory". We are interested in the pure electromagnetic effects upon gravity. It is not necessary to use his formalism, which includes products of Ricci tensors and is similar in complexity to the Gauss-Bonnet terms in the string effective action. The point is that we know how to create strong EM-fields. Our approach is not only to explain but to construct and parallels the efforts to build wormholes and warp drives. We use, however, fields which are common and satisfy all energy conditions and not exotic matter or the Casimir effect. Second, we try to induce gravitational force without altering very much the metric. This is possible thanks to the 20 orders of magnitude collected in $c^2/2$ in Eq. (7). We have demonstrated that except the Newtonian limit, general relativity possesses also a Maxwellian limit. Root gravity is one way to do this, but there are others too.

We have given here several laboratory set-ups, where root gravity can be detected and general relativity tested once more, this time in its highly non-linear regime. The most promising seems to be the well-known capacitor and its magnetic analogue. The effect has not been discovered yet, because capacitors are used in low-voltage circuits and they are first firmly fixed and charged afterwards. Small root gravity effects should be present also in today's solenoids with strong magnetic fields. The acceleration is masked by that of the Earth, being a fraction of it, and is hardly measurable because the working space is very small.

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